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ADAPTIVE STABILIZATION FOR CONTINUOUS TIME SYSTEMS WITH DISTURBANCES

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SUMMARY

This paper concerns adaptive stabilization for single-input/single-output (SISO) continuous time systems with unknown coefficients and containing stochastic or deterministic disturbances. The conditions used here are possibly the weakest: neither the positive realness condition nor the availability of the upper bound of system disturbances is needed; the only condition imposed on the system structure is stabilizability, which is necessary for stabilizing a system even in the case where the system coefficients are known. The adaptive control given in the paper is switched at stopping times either on external excitations or on certainty equivalence controls defined by the pole assignment method at fixed times. It is shown that after a finite period of time the external excitation is no longer used and the system is stabilized in the long run average sense.

KEY WORDS Adaptive stabilization Stochastic system Continuous time Stopping time Pole placement

1. INTRODUCTION

For the last two decades much attention has been paid to adaptive stabilization of both discrete time (see e.g. References 1–10) and continuous time (see e.g. References 11–21) stochastic and deterministic systems. Authors of previous papers, in addition to the stabilizability assumption which is necessary for the problem in question, require various extra conditions. For example, input strict passitivity and *a priori* knowledge on the location of the parameters are required in Reference 19; a lower bound for the coprimeness degree is needed in References 5, 7, 8 and 19; a location restriction on the unstable zeros is used in Reference 11; the minimum phase condition and the strictly positive realness condition on the transfer function of the system noise are applied in Reference 17; some conditions on the system input or output are needed in References 18 and 20; and CB is assumed known in Reference 21, where B and C are matrix coefficients for the system input and output respectively.

The purpose of this paper is to remove all extra restrictions on the system structure, i.e. to adaptively stabilize a system under the stabilizability assumption only. Let us explain this more precisely.

Let S be the integral operator

$$Sy_t = \int_0^t y_s \, ds$$

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and let the SISO continuous time stochastic system be described by

$$A(S)y_t = y_0 + SB(S)u_t + C(S)w_t + S\eta_t \quad \forall t \geqslant 0 \tag{1}$$

where $A(S)$, $B(S)$ and $C(S)$ are polynomials in S with unknown coefficients but known orders:

$$A(S) = 1 + \sum_{i=1}^{p} a_i S^i, \qquad B(S) = \sum_{i=1}^{q} b_i S^{i-1}, \qquad C(S) = \sum_{i=0}^{l} c_i S^i \tag{2}$$

In (1), $\{w_t, \mathscr{F}_t\}$ is a standard Wiener process with respect to a non-decreasing and right-continuous σ-algebra $\{\mathscr{F}_t\}$ defined on a probability space $(\Omega, P, \mathscr{F})$, y_0 is the initial value, $\mathscr{F}_t$-adapted y_t and u_t (i.e. y_t and u_t are $\mathscr{F}_t$-measurable,) are the system output and input respectively and $\mathscr{F}_t$-adapted η_t is the system disturbance, which is different from that driven by the Wiener process and may be deterministic.

When $l = p$ and $c_p = ga_p$, system (1) has the state space representation

$$\mathrm{d}x_t = Ax_t\, \mathrm{d}t + Bu_t\, \mathrm{d}t + C\, \mathrm{d}w_t + D\eta_t\, \mathrm{d}t \tag{3}$$

$$\mathrm{d}y_t = D^{\mathrm{T}}x_t\, \mathrm{d}t + g\, \mathrm{d}w_t \tag{4}$$

with

$$A = \begin{bmatrix} -a_1 & 1 & & \\ -a_2 & & \ddots & \\ \vdots & & & 1 \\ -a_m & & & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \qquad C = \begin{bmatrix} c_0 - ga_0 \\ \vdots \\ c_{m-1} - ga_{m-1} \end{bmatrix}, \qquad D \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Bigg\} m \tag{5}$$

where $m = \max\{p, q\}$, $a_0 = 1$, $a_i = 0$ for $i > p$, $b_j = 0$ for $j > q$, $c_k = 0$ for $k > l$ and X^{T} denotes the transpose of a vector or a matrix X.

When $C(S) = 0$ and η_t is deterministic, system (1) turns out to be a deterministic one:

$$\dot{x}_t = Ax_t + Bu_t + D\eta_t, \qquad y_t = y_0 + \int_0^t D^{\mathrm{T}}x_s\, \mathrm{d}s$$

Let us denote the collection of unknown coefficients of $A(S)$ and $B(S)$ by θ:

$$\theta = [-a_1, \ldots, -a_p, b_1, \ldots, b_q]^{\mathrm{T}} \tag{6}$$

For θ we use the least squares (LS) estimate θ_t which is defined as (see e.g. References 17, 19, 20 and 22)

$$\mathrm{d}\theta_t = R_t\varphi_t(\mathrm{d}y_t - \varphi_t^{\mathrm{T}}\theta_t\, \mathrm{d}t) \quad \text{with} \quad R_t = \left(I + \int_0^t \varphi_s\varphi_s^{\mathrm{T}}\, \mathrm{d}s\right)^{-1} \tag{7}$$

and

$$\varphi_t^{\mathrm{T}} = [y_t, \ldots, S^{p-1}y_t, u_t, \ldots, S^{q-1}u_t] \tag{8}$$

where $\theta_0 \in \mathscr{F}_0$ is arbitrarily chosen.

Based on θ_t, we want to design an adaptive control so that the system is stabilized under the following assumptions:

A1. $A(S)$ and $SB(S)$ are coprime, $b_q \neq 0$.

A2.
$$\sup_{t \geqslant 0} \frac{1}{t+1} \int_0^t \eta_s^2\, \mathrm{d}s < \infty \quad \text{a.s.}$$

Assumption A1 is not the weakest condition to stabilize a system for the case where θ is known, since in this case for stabilizability the necessary and sufficient condition is that the greatest common factor of $A(S)$ and $SB(S)$ be unity or a stable polynomial. However, in Lemma 6 in Appendix I it is shown that Assumptions A1 and A2 together are actually equivalent to the following.

A1′. The greatest common factor of $A(S)$ and $SB(S)$ is unity or a stable polynomial, $b_q \neq 0$, i.e. the system is stabilizable.

A2′.
$$\sup_{t \geqslant 0} \frac{1}{t+1} \int_0^t \eta_s^2 \, \mathrm{d}s < \infty \quad \text{a.s.}$$

Therefore, without loss of generality, in the sequel we will use Assumptions A1 and A2 directly.

2. ADAPTIVE CONTROL

To define the adaptive control, we need the certainty equivalence control, the external excitation and the stopping times where the switches of control take place. This will be completed after several lemmas.

Lemma 1[17]

Let $k \geqslant 0$ be an integer and $E(S) = 1 + e_1 S + \cdots + e_k S^k$ with $e_k \neq 0$ be a stable polynomial, i.e. $E(z) \neq 0$ for any z with $\mathrm{Re}(z) \geqslant 0$, where $\mathrm{Re}(z)$ denotes the real part of a complex number z. Then there is a constant $\mu_e \geqslant 1$ (depending on $E(S)$ only) such that

$$\sum_{i=0}^{k} \int_0^t \left(\frac{S^i}{E(S)} x_\lambda \right)^2 \mathrm{d}\lambda \leqslant \mu_e \int_0^t x_\lambda^2 \, \mathrm{d}\lambda$$

for any square-integrable process $\{x_t\}$.

For the proof we refer to Reference 17.

If $A(S)$ and $SB(S)$ are coprime and $b_q \neq 0$, then for any polynomial

$$E(S) = 1 + e_1 S + \cdots + e_{p+q} S^{p+q} \quad \text{with} \quad e_{p+q} \neq 0 \tag{9}$$

there exists a unique pair of polynomials $(G(S), H(S))$ such that

$$A(S)G(S) - SB(S)H(S) = E(S) \quad \text{with} \quad \partial(G(S)) \leqslant q-1 \quad \text{and} \quad \partial(H(S)) = p \tag{10}$$

where here and hereafter $\partial(X(S))$ denotes the degree of polynomial $X(S)$ in S.

From (10) and (1) it is clear that

$$\begin{aligned} E(S)y_t &= A(S)G(S)y_t - SB(S)H(S)y_t \\ &= G(s)[A(S)y_t - SB(S)u_t] + SB(S)[G(S)u_t - H(S)y_t] \\ &= G(S)[y_0 + C(S)w_t + S\eta_t] + SB(S)[G(S)u_t - H(S)y_t] \end{aligned} \tag{11}$$

and

$$\begin{aligned} E(S)u_t &= A(S)G(S)u_t - SB(S)H(S)u_t \\ &= H(s)[A(S)y_t - SB(S)u_t] + A(S)[G(S)u_t - H(S)y_t] \\ &= H(S)[y_0 + C(S)w_t + S\eta_t] + A(S)[G(S)u_t - H(S)y_t] \end{aligned} \tag{12}$$

Noticing that $\partial(G(S)) \leqslant q-1$, from (11) and Lemma 1 we see that in the case where θ is known, if $\partial(C(S)) \leqslant p$, $E(S)$ is stable and the control u_t is defined by

$$G(S)u_t - H(S)y_t = 0, \quad t \geqslant 0 \tag{13}$$

then under Assumptions A1 and A2 the system output is bounded in the average sense, i.e.

$$\sup_{t\geqslant 0} \frac{1}{t+1}\int_0^t y_s^2 \, \mathrm{d}s < \infty \quad \text{a.s.} \tag{14}$$

Similarly, from (12), $\partial(H(S)) = p$ and Lemma 1 it is clear that under Assumptions A1 and A2 the control u_t defined by (13) is bounded in the average sense, i.e.

$$\sup_{t\geqslant 0} \frac{1}{t+1}\int_0^t u_s^2 \, \mathrm{d}s < \infty \quad \text{a.s.} \tag{15}$$

if $\partial(C(S)) \leqslant q-1$ and $E(S)$ is stable.

Therefore, in the case where θ is known, if $\partial(C(S)) \leqslant \min\{p, q-1\}$ and Assumptions A1 and A2 hold, then for any stable $E(S)$ the control defined by (13) stabilizes the system, i.e.

$$\sup_{t\geqslant 0} \frac{1}{t+1}\int_0^t (y_s^2 + u_s^2) \, \mathrm{d}s < \infty \quad \text{a.s.} \tag{16}$$

Replacing a_i, $i = 1, \ldots, p$, and b_j, $j = 1, \ldots, q$, in $A(S)$ and $SB(S)$ respectively by their estimates a_{it} and b_{jt} given by θ_t, we denote the results by $A_t(S)$ and $SB_t(S)$. In the case where $A_t(S)$ and $SB_t(S)$ are coprime, in a similar way to (10) we can obtain a pair of polynomials $(G_t(S), H_t(S))$.

If θ_{t_1} is an 'accurate' estimate for θ, then the certainty equivalence control given at time t_1 will hopefully work for $t \geqslant t_1$. However, in general we have no reason to expect that $\theta_t - \theta$ is small.

In order to obtain a 'good' estimate θ_t, we will use the excitation technique. For this we give two lemmas.

Lemma 2

Let

$$r_t = 1 + \int_0^t \|\varphi_s\|^2 \, \mathrm{d}s$$

and let $\lambda_{\min}^{(t)}$ denote the smallest eigenvalue of the matrix R_t^{-1}, where φ_t and R_t are as given by (8) and (7) respectively. Then the parameter estimate θ_t given by (7) and (8) has the property

$$\|\theta_t - \theta\|^2 \leqslant \frac{\varkappa[(t+1)^{2l+1} + \log r_t]}{\lambda_{\min}^{(t)}} \quad \text{a.s.} \quad \forall t \geqslant 0$$

where $l = \partial(C(s))$ and $\varkappa$ is a random variable independent of time t.

The proof is given in Appendix II.

Lemma 3

Let α be an arbitrary positive constant. Define for $i = 1, 2, \ldots, p+q$

$$\beta_i = (-1)^{i+1}\alpha^i \frac{(p+q)!}{i! \times (p+q-i)!} \quad \text{with} \quad 0! \triangleq 1 \quad \text{and} \quad i! \triangleq 1 \times 2 \times \cdots \times i \tag{17}$$

and for any $t \geqslant 0$

$$u_t' = 1 + \beta_1 S u_t' + \cdots + \beta_{p+q} S^{p+q} u_t' \quad \text{with} \quad u_0' = 1 \tag{18}$$

If $u_t = u_t'$ for $t \geqslant \tau$ in (1), where τ is a given stopping time, then Assumptions A1 and A2 imply that there exist $T \geqslant \tau$, $\rho > 0$, $\rho' \geqslant 0$ and $a > b > 1$ such that

$$\lambda_{\min}^{(t)} \geqslant \rho b^t \quad \text{and} \quad r_t \leqslant \rho' a^t \quad \forall t \geqslant T \tag{19}$$

The proof is given in Appendix III.

Let $R(S)$ be a stable polynominal in S and let ξ_t^u and ξ_t^y denote the filtered values of u_t and y_t respectively, i.e.

$$R(S)\xi_t^u = u_t \quad \text{and} \quad R(S)\xi_t^y = y_t \quad \forall t \geqslant 0 \tag{20}$$

Set

$$\zeta_t = [S\xi_t^y, \ \ldots, \ S^p \xi_t^y, \ S\xi_t^u, \ \ldots, \ S^q \xi_t^u]^{\mathrm{T}}$$

It is easy to see that

$$R(S)\zeta_t = S\varphi_t \tag{21}$$

Arbitrarily choose a deterministic sequence $\{\varepsilon_t\}$ such that

$$0 < \varepsilon_t < 1, \qquad \varepsilon_t \to 0, \qquad \varepsilon_t(t+1) \to \infty \tag{22}$$

In what follows, by the norm of a polynomial

$$X(S) = \sum_{i=0}^{\partial(X(S))} x_i S^i$$

we mean

$$\| X(S) \| = \left(\sum_{i=0}^{\partial(X(S))} \| x_i \|^2 \right)^{1/2}$$

We now define two sequences of stopping times $\{\tau_i\}$ and $\{\sigma_i\}$ as follows:

$$0 = \tau_0 < \sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \cdots$$

$$\begin{aligned}
\sigma_i = \inf\Big\{ t \geqslant \tau_{i-1} + 1 \colon & \int_0^t \varphi_s \varphi_s^{\mathrm{T}} \, \mathrm{d}s \geqslant [(t+1)^{2\varrho} + t \log r_t] \varepsilon_t^{-2} I; \\
& A_t(S) G_t(S) - S B_t(S) H_t(S) = E(S) \\
& \text{is solvable with respect to } G_t(S) \text{ and } H_t(S) \text{ subject to} \\
& \partial(G_t(S)) \leqslant q - 1 \quad \text{and} \quad \partial(H_t(S)) = p; \\
& \| G_t(S) \|^2 + \| H_t(S) \|^2 \leqslant \frac{1}{2\mu_e (p+q+1)\varepsilon_t}; \\
& \int_0^t (\xi_s^y - \theta_t^{\mathrm{T}} \zeta_s)^2 \, \mathrm{d}s \leqslant \varepsilon_t^2 \Gamma_t((t_i+1)^2) \Big\}
\end{aligned} \tag{23}$$

$$\tau_i = \inf\left\{ t \geqslant \sigma_i + 1 \colon \int_0^t (\xi_s^y - \theta_{\sigma_i}^{\mathrm{T}} \zeta_s)^2 \, \mathrm{d}s > \varepsilon_{\sigma_i}^2 \Gamma_t((\sigma_i + 1)^2) \right\} \tag{24}$$

where $\varrho = \partial(R(S))$, r_t is as defined in Lemma 2, $E(S)$ is a stable polynomial in S given by (9),

μ_e is the constant appearing in Lemma 1 when $k = p + q$, and

$$\Gamma_t(x) = (t+1) \sup_{0 \leqslant \lambda \leqslant t} \left\{ x + \frac{1}{\lambda + 1} \int_0^\lambda \left(\sum_{j=0}^{p+1} (S^j \xi_s^y)^2 + \sum_{j=0}^{q} (S^j \xi_s^u)^2 \right) ds \right\} \tag{25}$$

Finally we define the adaptive control u_t as

$$u_t = \begin{cases} u_t' & \text{if } t \in (\tau_i, \sigma_{i+1}] \text{ for some } i \geqslant 0 \\ H_{\sigma_i}(S) y_t - [G_{\sigma_i}(S) - 1] u_t, & \text{if } t \in (\sigma_i, \tau_i] \text{ for some } i \geqslant 1 \end{cases} \tag{26}$$

where u_t' is as given in Lemma 3.

From (23)–(26) we see that the mechanism of the adaptive control (26) is similar to that in Reference 9: if the accuracy of the parameter estimate is not satisfactory, then we use the external excitation u_t' to make the LS estimate more accurate; if the parameter estimate is acceptable, then we use the certainty equivalence control to stabilize the system.

Since the upper bounds for $\| G(S) \|^2 + \| H(S) \|^2$,

$$\int_0^t \eta_s^2 \, ds \quad \text{and} \quad \int_0^t [R^{-1}(S) C(S) w_s]^2 \, ds$$

are unknown, we include a multiple $1/\varepsilon_t$ in the last but one inequality of (23) and put $(t+1)^2$ in $\Gamma_t(\cdot)$ in the last inequality of (23) in order to guarantee $\sigma_i < \infty$.

It is natural that the complexity of an adaptive control depends upon the *a priori* knowledge about the system structure and system disturbances. The less the *a priori* knowledge is, the more complex the adaptive control is. When the system disturbance $C(S) w_t$ or η_t does exist, it seems unavoidable to apply a rather complicated control similar to that given by (26) (see e.g. References 9 and 16) to guarantee the stability of the closed-loop system.

It is worth noticing that the excitation signal used in (26) is different from those used in References 9 and 16: u_t' in (26) is deterministic and independent of system (1) (see (28) below). As shown in Lemma 3 the excitation u_t' in (26) diverges to infinity as t goes to infinity, but Lemma 5 below proves that u_t' is actually used only for a finite period of time. Excitation and switching techniques for discrete time adaptive control systems are described in detail in References 23 and 24.

We now consider the solvability of (1), (7) and (26) for y_t, θ_t and u_t.

Lemma 4

Assume that the system disturbance η_t does not affect the solvability of the closed-loop system. Then the system consisting of (1), (7) and (26) has a unique solution (y_t, θ_t, u_t).

Proof. Consider

$$A(S) y_t = y_0 + SB(S) u_t + C(S) w_t \tag{27}$$

From the assumption of the lemma it suffices to show that the system consisting of (27), (7) and (26) has a unique solution (y_t, u_t, θ_t).

From (17), (18) and (51) in Appendix III it follows that

$$u_t' = H^{\mathrm{T}} e^{\Lambda t} H \quad \forall t \geqslant 0 \tag{28}$$

where H and Λ are as defined by (50) in Appendix III.

Hence u_t' is uniquely defined for all $t \geqslant 0$, no matter what adaptive control is applied.

Substituting (28) into (27), we get

$$A(S)y_t = y_0 + SB(S)(H^{\mathrm{T}}e^{\Lambda t}H) + C(S)w_t \tag{29}$$

which has a unique solution y_t for all $t \geqslant 0$.

Thus φ_t and R_t in (7) and (8) are well defined for all $t \geqslant 0$. This in turn implies that equation (7) is a linear stochastic differential equation with continuous coefficients in the interval $[0, T]$ for any given $T \geqslant 1$. Thus equation (7) has a unique solution θ_t which is continuous in $[0, T]$ for any given $T \geqslant 1$.

Therefore σ_1 can be defined from (23). Furthermore, σ_1 is a stopping time, since by definition (23) $\sigma_1 \geqslant 1$ and

$$\{\omega\colon\ \sigma_1 \leqslant t\} = \{\omega\colon\ \sigma_1 > t\}^c \in \mathscr{F}_{t^+} = \mathscr{F}_t \quad \forall t \geqslant 1$$

If $\sigma_1 < \infty$, then from (26) and (27) it follows that

$$\begin{bmatrix} A(S) & -SB(S) \\ -H_{\sigma_1}(S) & G_{\sigma_1}(S) \end{bmatrix} \begin{bmatrix} y_t \\ u_t \end{bmatrix} = \begin{bmatrix} y_0 + C(S)w_t \\ 0 \end{bmatrix} \quad \forall t > \sigma_1 \tag{30}$$

This is a linear stochastic differential equation with time-independent coefficients.

Noticing that $A(S)G_{\sigma_1}(S) - SB(S)H_{\sigma_1}(S) = 1$ at $S = 0$, we see that $A(S)G_{\sigma_1}(S) - SB(S)H_{\sigma_1}(S)$ is not identically zero, so (30) has a unique solution (y_t, u_t) for all $t \geqslant \sigma_1$. This in turn implies that (7) has a unique, continuous solution θ_t in the interval $[\sigma_1, T]$ for any given $T \geqslant \sigma_1 + 1$.

Thus τ_1 can be defined from (24). Furthermore, τ_1 is a stopping time, since by definition (24) $\tau_1 \geqslant \sigma_1 + 1$ and $\forall t \geqslant \sigma_1 + 1$

$$\{\omega\colon\ \tau_1 \leqslant t\} = \Omega - \{\omega\colon\ \tau_1 \geqslant t\} = \left\{\omega\colon\ \int_0^\lambda (\xi_s^y - \theta_{\sigma_1}^{\mathrm{T}}\zeta_s)^2\,\mathrm{d}s \leqslant \varepsilon_{\sigma_1}^2\Gamma_\lambda((\sigma_1+1)^2),\ \sigma_1 + 1 \leqslant \lambda \leqslant t\right\}^c \in \mathscr{F}_t$$

Repeating the same argument, we see that the system consisting of (27), (26) and (7) has a unique, continuous solution (y_t, u_t, θ_t) in $[0, T]$ for any given $T > 0$. Q.E.D.

3. MAIN RESULTS

We now formulate and prove our main results of this paper.

Lemma 5

Suppose that Assumptions A1 and A2 hold and the system disturbance η_t does not affect the solvability of the closed-loop system. If the stable polynomial $R(S)$ in (20) satisfies $\partial(R(S)) \geqslant \partial(C(S)) + 1$, then under the adaptive control (26) there exists a positive integer-valued random variable i such that $\sigma_i < \infty$ and $\tau_i = \infty$ a.s., where $\{\tau_i\}$ and $\{\sigma_i\}$ are as defined by (23) and (24) respectively.

The proof is given in Appendix IV.

Theorem 1

Under the conditions of Lemma 5 the adaptive control (26) stabilizes the closed-loop system (1), (7) and (26) in the following sense:

$$\limsup_{t\to\infty} \frac{1}{t+1} \int_0^t \left(\sum_{j=0}^{p+1} (S^j\xi_s^y)^2 + \sum_{j=0}^{q} (S^j\xi_s^u)^2 \right) \mathrm{d}s < \infty \quad \text{a.s.} \tag{31}$$

where ξ_t^y and ξ_t^u are as given by (20).

Proof. By Lemma 5 we know that there exists a positive integer-valued random variable i such that $\sigma_i < \infty$ and $\tau_i = \infty$ a.s. Hence from (26) it follows that

$$H_{\sigma_i}(S)y_t - G_{\sigma_i}(S)u_t = 0 \quad \text{a.s.} \quad \forall t \geqslant \sigma_i \tag{32}$$

Noticing (23) we obtain

$$\begin{aligned} E(S)S^j y_t &= S^j A_{\sigma_i}(S) G_{\sigma_i}(S) y_t - S^{j+1} B_{\sigma_i}(S) H_{\sigma_i}(S) y_t \\ &= S^j G_{\sigma_i}(S)[A_{\sigma_i}(S)y_t - SB_{\sigma_i}(S)u_t] \\ &\quad + S^{j+1}B_{\sigma_i}(S)[G_{\sigma_i}(S)u_t - H_{\sigma_i}(S)y_t], \quad j = 0, 1, \ldots, p+1 \end{aligned} \tag{33}$$

and

$$\begin{aligned} E(S)S^j u_t &= S^j H_{\sigma_i}(S)[A_{\sigma_i}(S)y_t - SB_{\sigma_i}(S)u_t] \\ &\quad + S^j A_{\sigma_i}(S)[G_{\sigma_i}(S)u_t - H_{\sigma_i}(S)y_t], \quad j = 0, 1, \ldots, q \end{aligned} \tag{34}$$

Taking into account that $A_{\sigma_i}(S)y_t - SB_{\sigma_i}(S)u_t = y_t - \theta_{\sigma_i}^{\mathrm{T}} S\varphi_t$, by (33) we have

$$\begin{aligned} \sum_{j=0}^{p+1} (S^j \xi_s^y)^2 &\leqslant 2\|G_{\sigma_i}(S)\|^2 \sum_{j=0}^{p+1} \sum_{k=0}^{q-1} [S^{j+k}E^{-1}(S)(\xi_s^y - \theta_{\sigma_i}^{\mathrm{T}}\zeta_s)]^2 \\ &\quad + 2\sum_{j=0}^{p+1} \{E^{-1}(S)R^{-1}(S)S^{j+1}B_{\sigma_i}(S)[G_{\sigma_i}(S)u_s - H_{\sigma_i}(S)y_s]\}^2 \\ &\leqslant 2\|G_{\sigma_i}(S)\|^2(p+q+1) \sum_{j=0}^{p+q} [S^j E^{-1}(S)(\xi_s^y - \theta_{\sigma_i}^{\mathrm{T}}\zeta_s)]^2 \\ &\quad + 2\sum_{j=0}^{p+1} \{E^{-1}(S)R^{-1}(S)S^{j+1}B_{\sigma_i}(S)[G_{\sigma_i}(S)u_s - H_{\sigma_i}(S)y_s]\}^2 \end{aligned} \tag{35}$$

and similarly by (34) we obtain

$$\begin{aligned} \sum_{j=0}^{q} (S^j \xi_s^u)^2 &\leqslant 2\|H_{\sigma_i}(S)\|^2 \sum_{j=0}^{q} \sum_{k=0}^{q} [S^{j+k}E^{-1}(S)(\xi_s^y - \theta_{\sigma_i}^{\mathrm{T}}\zeta_s)]^2 \\ &\quad + 2\sum_{j=0}^{q} \{E^{-1}(S)R^{-1}(S)S^{j}A_{\sigma_i}(S)[G_{\sigma_i}(S)u_s - H_{\sigma_i}(S)y_s]\}^2 \\ &\leqslant 2\|H_{\sigma_i}(S)\|^2(p+q+1) \sum_{j=0}^{p+q} [S^j E^{-1}(S)(\xi_s^y - \theta_{\sigma_i}^{\mathrm{T}}\zeta_s)]^2 \\ &\quad + 2\sum_{j=0}^{q} \{E^{-1}(S)R^{-1}(S)S^{j}A_{\sigma_i}(S)[G_{\sigma_i}(S)u_s - H_{\sigma_i}(S)y_s]\}^2 \end{aligned} \tag{36}$$

By Lemma 1 and (32) we see that

$$\frac{1}{t+1}\sum_{j=0}^{p+1}\int_0^t \{E^{-1}(S)R^{-1}(S)S^{j+1}B_{\sigma_i}(S)[G_{\sigma_i}(S)u_s - H_{\sigma_i}(S)y_s]\}^2\, ds < \infty \quad \text{a.s.}$$

and

$$\frac{1}{t+1}\sum_{j=0}^{q}\int_0^t \{E^{-1}(S)R^{-1}(S)S^{j}A_{\sigma_i}(S)[G_{\sigma_i}(S)u_s - H_{\sigma_i}(S)y_s]\}^2\, ds < \infty \quad \text{a.s.}$$

Therefore by (35), (36) and Lemma 1 we conclude that for some $\nu_1 < \infty$, which is independent of t,

$$\frac{1}{t+1}\int_0^t\left(\sum_{k=0}^{p+1}(S^k\xi_s^y)^2+\sum_{k=0}^{q}(S^k\xi_s^u)^2\right)\mathrm{d}s$$

$$\leqslant 2(p+q+1)[\|G_{\sigma_i}(S)\|^2+\|H_{\sigma_i}(S)\|^2]\frac{1}{t+1}\sum_{j=0}^{p+q}\int_0^t\left(\frac{S^j}{E(S)}(\xi_s^y-\theta_{\sigma_i}^{\mathrm{T}}\zeta_s)\right)^2\mathrm{d}s+\nu_1$$

$$\leqslant\frac{2(p+q+1)}{2\mu_e(p+q+1)\varepsilon_{\sigma_i}}\frac{\mu_e}{t+1}\int_0^t(\xi_s^y-\theta_{\sigma_i}^{\mathrm{T}}\zeta_s)^2\,\mathrm{d}s+\nu_1$$

$$\leqslant\varepsilon_{\sigma_i}\frac{1}{t+1}\Gamma_t((\sigma_i+1)^2)+\nu_1\quad\text{a.s.},\quad t\geqslant\sigma_i+1 \tag{37}$$

where (24), $\sigma_i<\infty$ and $\tau_i=\infty$ a.s. have been used for the last inequality.

Set

$$\nu_2=\nu_1+\sup_{0\leqslant\lambda\leqslant\sigma_i+1}\left\{\frac{1}{\lambda+1}\int_0^\lambda\left(\sum_{k=0}^{p+1}(S^k\xi_s^y)^2+\sum_{k=0}^{q}(S^k\xi_s^u)^2\right)\mathrm{d}s\right\}$$

Then from (37) and (25) it follows that

$$\sup_{0\leqslant\lambda\leqslant t}\left\{\frac{1}{\lambda+1}\int_0^\lambda\left(\sum_{k=0}^{p+1}(S^k\xi_s^y)^2+\sum_{k=0}^{q}(S^k\xi_s^u)^2\right)\mathrm{d}s\right\}\leqslant\varepsilon_{\sigma_i}\frac{1}{t+1}\Gamma_t((\sigma_i+1)^2)+\nu_2$$

$$\leqslant\varepsilon_{\sigma_i}(\sigma_i+1)^2+\nu_2+\varepsilon_{\sigma_i}\sup_{0\leqslant\lambda\leqslant t}\left\{\frac{1}{\lambda+1}\int_0^\lambda\left(\sum_{k=0}^{p+1}(S^k\xi_s^y)^2+\sum_{k=0}^{q}(S^k\xi_s^u)^2\right)\mathrm{d}s\right\}\quad\text{a.s.}$$

i.e.

$$\sup_{0\leqslant\lambda\leqslant t}\left\{\frac{1}{\lambda+1}\int_0^\lambda\left(\sum_{k=0}^{p+1}(S^k\xi_s^y)^2+\sum_{k=0}^{q}(S^k\xi_s^u)^2\right)\mathrm{d}s\right\}\leqslant(1-\varepsilon_{\sigma_i})^{-1}[\nu_2+\varepsilon_{\sigma_i}(\sigma_i+1)^2]<\infty\quad\text{a.s.}$$

which implies (31). Q.E.D.

Corollary 1

Suppose $\partial(C(S))\leqslant p$ in addition to the conditions of Theorem 1. Then for any given stable polynomial $R(S)$ with $\partial(R(S))=p+1$ the adaptive control (26) leads to both (31) and (14).

Proof. Following the argument of Theorem 1 and noticing that $\partial(C(S))\leqslant p$ and $\partial(R(S))=p+1$ imply

$$\int_0^t\left(\frac{C(S)}{R(S)}w_s\right)^2\mathrm{d}s=O(t)\quad\text{a.s.}$$

we conclude that (31) is still true.

Rewrite $R(S)$ as

$$R(S)=\sum_{j=0}^{p+1}r_j'S^j$$

Then from (20) it follows that

$$\limsup_{t\to\infty} \frac{1}{t+1}\int_0^t y_s^2\,\mathrm{d}s \leqslant \|R(S)\|^2 \limsup_{t\to\infty}\frac{1}{t+1}\int_0^t \sum_{j=0}^{p+1}(S^j\xi_s^y)^2\,\mathrm{d}s < \infty \quad \text{a.s.}$$

where (31) is invoked for the last inequality. Q.E.D.

Similarly from Theorem 1 we have the following corollaries.

Corollary 2

If $\partial(C(S)) \leqslant q-1$ and the conditions of Theorem 1 are satisfied, then for any given stable polynomial $R(S)$ with $\partial(R(S))=q$ the adaptive control (26) leads to both (31) and (15).

Corollary 3

If $\partial(C(S)) \leqslant \min\{q-1,p\}$ and the conditions of Theorem 1 hold, then for any given stable polynomial $R(S)$ with $\partial(R(S))=\min\{p+1,q\}$ the adaptive control (26) leads to both (31) and (16).

4. CONCLUSIONS

This paper deals with adaptive stabilization for SISO continuous time linear systems disturbed by both purely random noise and a function η_t which may characterize the deterministic disturbance. The only requirement for η_t is the boundedness of its time average. Systems are adaptively stabilized under the stabilizability assumption only.

The adaptive control is switched at stopping times either on the certainty equivalence control or on an external excitation, which as is shown is used only for a finite period of time.

To conclude, we would like to point out that the behaviour of the adaptive control system with the certainty equivalence control applied without external excitation is not clear even if $\eta_t \equiv 0$.

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APPENDIX I

Lemma 6

Suppose that the greatest common factor of $A(S)$ and $SB(S)$ is a stable polynomial

$$F(S) = 1 + \sum_{i=1}^{r} f_i S^i \quad \text{with} \quad f_r \neq 0.$$

If $r \geqslant 1$ is known and Assumptions A1 and A2 hold, then system (1) is equivalent to

$$A'(S)y_t = y_0 + SB'(S)u_t + C'(S)w_t + S\eta_t' \quad \forall t \geqslant 0 \tag{38}$$

where $A'(S)$, $B'(S)$ and $C'(S)$ are polynomials in S,

$$A'(S) = 1 + \sum_{i=1}^{p-r} a_i' S^i, \qquad B'(S) = \sum_{i=1}^{q-r} b_i' S^{i-1}, \qquad C'(S) = \sum_{i=0}^{l-r} c_i' S^i \tag{39}$$

and $A'(S)$, $B'(S)$ and η_t' satisfy the following statements

(I) $A'(S)$ and $SB'(S)$ are coprime, $b_{q-r}' \neq 0$, and $p-r$ and $q-r$ are known.

(II)
$$\sup_{t \geqslant 0} \frac{1}{t+1} \int_0^t (\eta_s')^2 \, \mathrm{d}s < \infty \quad \text{a.s.}$$

Proof. Let

$$A'(S) = F^{-1}(S)A(S), \qquad B'(S) = F^{-1}(S)SB(S) \tag{40}$$

The first two equations of (39) and all assertions of Statement (I) immediately follow from the fact that $b_q \neq 0$, $f_r \neq 0$ and p, q and r are known.

We now find $C'(S)$ of degree $l-r$ and η_t' such that (38) and Statement II hold.

Let v_t be the solution of the integral equation

$$F(S)v_t = w_t, \quad t \geqslant 0$$

and let

$$M_F = \begin{bmatrix} -f_1 & \cdots & -f_{r-1} & -f_r \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \qquad V_t = \begin{bmatrix} v_t \\ Sv_t \\ \vdots \\ S^{r-1}v_t \end{bmatrix}, \qquad D_1 = \left.\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} r$$

Then we have

$$\mathrm{d}V_t = M_F V_t \, \mathrm{d}t + D_1 \, \mathrm{d}w_t \quad \text{and} \quad v_t = D_1^{\mathrm{T}} V_t \quad t \geqslant 0 \tag{41}$$

Since M_F is stable, by Reference 17 we see that

$$\frac{1}{t} \int^t V_\lambda V_\lambda^{\mathrm{T}} \, \mathrm{d}\lambda \xrightarrow[t \to \infty]{} \int_0^\infty \mathrm{e}^{M_F \lambda} D_1 D_1^{\mathrm{T}} \mathrm{e}^{M_F^{\mathrm{T}} \lambda} \, \mathrm{d}\lambda \quad \text{a.s.}$$

which together with (41) implies that

$$v_t = w_t + S(D_1^{\mathrm{T}} M_F V_t) \quad \text{with} \quad \sup_{t \geqslant 0} \frac{1}{t+1} \int^t \| V_\lambda \|^2 \, \mathrm{d}\lambda < \infty \quad \text{a.s.} \tag{42}$$

Following the argument of (28), we see that if z_t is the solution of $F(S)z_t = y_0$ for all $t \geqslant 0$, then

$$z_t = D_1^{\mathrm{T}} \mathrm{e}^{M_F t} D_1 y_0 = y_0 + S(D_1^{\mathrm{T}} \mathrm{e}^{M_F t} M_F D_1 y_0)$$

Let $C''(S) = \sum_{i=0}^{l-r-1} c_i'' S^i$ and $L(S)$ be the the unique solution of

$$\sum_{i=1}^{l} c_i S^{i-1} = F(S)C''(S) + L(S) \quad \text{with} \quad \partial(L(S)) \leqslant r-1$$

and let

$$C'(S) = SC''(S) + c_0 = c_0 + \sum_{i=1}^{l-r} c_{i-1}'' S^i \tag{43}$$

Then from the first assertion of (42) and the fact that $w_0 = 0$ and $v_t = F^{-1}(S)w_t$ it follows that

$$\begin{aligned} F^{-1}(S)C(S)w_t &= SC''(S)w_t + SF^{-1}(S)L(S)w_t + c_0 F^{-1}(S)w_t \\ &= SC''(S)w_t + SF^{-1}(S)L(S)w_t + c_0[w_t + S(D_1^{\mathrm{T}} M_F V_t)] \\ &= C'(S)w_t + SF^{-1}(S)L(S)w_t + S(c_0 D_1^{\mathrm{T}} M_F V_t) \end{aligned}$$

Therefore by Lemma 1, stability of M_F and the second assertion of (42) we have

$$\sup_{t \geqslant 0} \frac{1}{t+1} \int_0^t (\eta_s')^2 \, \mathrm{d}s < \infty \quad \text{a.s.} \tag{44}$$

where

$$\eta_t' = F^{-1}(S)\eta_t + F^{-1}(S)L(S)w_t + c_0 D_1^{\mathrm{T}} M_F V_t + D_1^{\mathrm{T}} e^{M_F t} M_F D_1 y_0$$

Finally, multiplying $F^{-1}(S)$ on both sides of (1) leads to

$$A'(S)y_t = y_0 + SB'(S)u_t + C'(S)w_t + S\eta_t'$$

which combined with (43) and (44) tells us that (38), the last equation of (39) and Statement (II) are true. Thus Lemma 6 holds. Q.E.D.

APPENDIX II

Proof of Lemma 2

Let $\tilde{\theta}_t = \theta_t - \theta$. Then from (1), (2), (6) and (8) it follows that

$$\mathrm{d}y_t = \theta^{\mathrm{T}}\varphi_t\,\mathrm{d}t + c_0\,\mathrm{d}w_t + \sum_{i=1}^{l} c_i S^{i-1} w_t\,\mathrm{d}t + \eta_t\,\mathrm{d}t$$

Substituting this into the first equation of (7) yields

$$\mathrm{d}\tilde{\theta}_t = -R_t\varphi_T\varphi_t^{\mathrm{T}}\tilde{\theta}_t\,\mathrm{d}t + R_t\varphi_t\left(c_0\,\mathrm{d}w_t + \sum_{i=1}^{l} c_i S^{i-1} w_t\,\mathrm{d}t + \eta_t\,\mathrm{d}t\right) \tag{45}$$

By this and the second equation of (7) we obtain

$$\mathrm{d}(\tilde{\theta}_t^{\mathrm{T}} R_t^{-1}\tilde{\theta}_t) = -(\tilde{\theta}_t^{\mathrm{T}}\varphi_t)^2\,\mathrm{d}t + c_0^2\varphi_t^{\mathrm{T}} R_t\varphi_t\,\mathrm{d}t + 2\tilde{\theta}_t^{\mathrm{T}}\varphi_t\left(c_0\,\mathrm{d}w_t + \sum_{i=1}^{l} c_i S^{i-1} w_t\,\mathrm{d}t + \eta_t\,\mathrm{d}t\right)$$

which implies that

$$0 \leqslant \tilde{\theta}_t^{\mathrm{T}} R_t^{-1}\tilde{\theta}_t \leqslant \tilde{\theta}_0^{\mathrm{T}} R_0^{-1}\tilde{\theta}_0 - \int_0^t (\tilde{\theta}_s^{\mathrm{T}}\varphi_s)^2\,\mathrm{d}s + c_0^2\int_0^t \varphi_s^{\mathrm{T}} R_s\varphi_s\,\mathrm{d}s$$
$$+ 2\int_0^t \tilde{\theta}_s^{\mathrm{T}}\varphi_s\left(c_0\,\mathrm{d}w_s + \sum_{i=1}^{l} c_i S^{i-1} w_s\,\mathrm{d}t + \eta_s\,\mathrm{d}s\right) \tag{46}$$

By Lemma 4 of Reference 25 we see that

$$2\int_0^t \tilde{\theta}_s^{\mathrm{T}}\varphi_s c_0\,\mathrm{d}w_t = O(1) + o\left(\left(\int_0^t (\tilde{\theta}_s^{\mathrm{T}}\varphi_s)^2\,\mathrm{d}s\right)^{3/4}\right) \quad \text{a.s.} \tag{47}$$

Noticing that by induction for any integer $i \geqslant 1$ and any integrable function f_t

$$S^i f_t = \int_0^t \frac{(t-s)^{i-1}}{(i-1)!} f_s\,\mathrm{d}s$$

we have

$$\int_0^t (S^i w_s)^2\,\mathrm{d}s \leqslant \int_0^t\int_0^\lambda \frac{(\lambda-s)^{2(i-1)}}{[(i-1)!]^2}\,\mathrm{d}s \int_0^\lambda w_s^2\,\mathrm{d}s\,\mathrm{d}\lambda$$
$$= \int_0^t \frac{t^{2i}-s^{2i}}{2i(2i-1)[(i-1)!]^2} w_s^2\,\mathrm{d}s \leqslant \varkappa_1(t+1)^{2i+3} \tag{48}$$

where $i = 0, 1, 2, \ldots$ and $\varkappa_1$ is random but independent of time t.

From (48) and Assumption A2 it follows that

$$2\int_0^t \tilde{\theta}_s^{\mathrm{T}}\varphi_s\left(\sum_{i=1}^{l} c_i S^{i-1} w_s + \eta_s\right)\mathrm{d}s \leqslant \frac{1}{2}\int_0^t (\tilde{\theta}_s^{\mathrm{T}}\varphi_s)^2\,\mathrm{d}s + 2\int_0^t\left(\sum_{i=1}^{l} c_i S^{i-1} w_s + \eta_s\right)^2\mathrm{d}s$$
$$\leqslant \frac{1}{2}\int_0^t (\tilde{\theta}_s^{\mathrm{T}}\varphi_s)^2\,\mathrm{d}s + \varkappa_2(t+1)^{2l+1} \quad \text{a.s.} \tag{49}$$

where $\varkappa_2$ is random but independent of time t.

Let tr(X) and det(X) denote the trace and determinant of a matrix X respectively. It is easy to see that

$$\int_0^t \varphi_s^{\mathrm{T}} R_s \varphi_s \,\mathrm{d}s = \mathrm{tr}\left(\int_0^t R_s \varphi_s \varphi_s^{\mathrm{T}} \,\mathrm{d}s\right) = \mathrm{tr}\left(\int_0^t R_s \,\mathrm{d}R_s^{-1}\right)$$

$$= \mathrm{tr}\left(\int_0^t \frac{\mathrm{d}(\det(R_s^{-1}))}{\det(R_s^{-1})}\right) \leqslant (p+q)\log r_t$$

Substituting this, (47) and (49) into (46) results in the desired result of Lemma 2. Q.E.D.

APPENDIX III

Proof of Lemma 3

Let

$$\Lambda = \begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_{p+q} \\ 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{bmatrix}, \qquad H = [1, \underbrace{0, \dots, 0}_{p+q-1}]^{\mathrm{T}} \tag{50}$$

and for any $t \geqslant 0$

$$U_t' = [u_t', Su_t', \dots, S^{p+q-1}u_t']^{\mathrm{T}}$$

Then from definition (18) it follows that for any $t > 0$

$$\frac{\mathrm{d}U_t'}{\mathrm{d}t} = \Lambda U_t' \quad \text{with} \quad U_0' = H \tag{51}$$

We first show that there exist constants $\rho_1 > 0$, $\gamma > 1$ and $T_1 \geqslant 0$ such that

$$\lambda_{\min}\left(\int_0^t U_s' U_s'^{\mathrm{T}} \,\mathrm{d}s\right) \geqslant \rho_1 \gamma^t \quad \forall t \geqslant T_1 \tag{52}$$

where here and hereafter $\lambda_{\min}(X)$ denotes the minimum eigenvalue of a matrix X.

From (17) and (50) it is easy to see that the characteristic polynomial $\det(xI - \Lambda) = (x-\alpha)^{p+q}$ of a matrix Λ coincides with the minimal polynomial of Λ. Thus there is a non-singular $(p+q)\times(p+q)$ matrix P such that

$$\bar{\Lambda} \triangleq P^{-1}\Lambda P = \begin{bmatrix} \alpha & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \alpha \end{bmatrix}_{(p+q)\times(p+q)} \tag{53}$$

Let $\bar{U}_t = P^{-1}U_t'$ and $\bar{H} = P^{-1}H$. Then (51) is equivalent to

$$\frac{\mathrm{d}\bar{U}_t}{\mathrm{d}t} = \bar{\Lambda}\bar{U}_t \quad \text{with} \quad \bar{U}_0 = \bar{H} \quad \forall t > 0 \tag{54}$$

Noticing that $\lambda_{\min}(PP^{\mathrm{T}}) > 0$ and

$$\lambda_{\min}\left(\int_0^t U_s' U_s'^{\mathrm{T}} \,\mathrm{d}s\right) \geqslant \lambda_{\min}(PP^{\mathrm{T}})\lambda_{\min}\left(\int_0^t \bar{U}_s \bar{U}_s^{\mathrm{T}} \,\mathrm{d}s\right) \quad \forall t \geqslant 0$$

we see that in order to show (52) it suffices to prove that there exist constants $\rho_1' > 0$, $\gamma > 1$ and $T_1 \geqslant 0$ such that

$$\lambda_{\min}\left(\int_0^t \bar{U}_s \bar{U}_s^{\mathrm{T}} \,\mathrm{d}s\right) \geqslant \rho_1' \gamma^t \quad \forall t \geqslant T_1 \tag{55}$$

From (54) it follows that

$$\bar{U}_t = \mathrm{e}^{\bar{\Lambda} t}\bar{H} \quad \forall t \geqslant 0$$

which implies that for any constant $\delta > 0$ and any time instant $t \geqslant \delta$

$$\int_0^t \bar{U}_s \bar{U}_s^{\mathrm{T}}\, \mathrm{d}s = \int_0^t \mathrm{e}^{\bar{\Lambda}s}\bar{H}.\bar{H}^{\mathrm{T}}\mathrm{e}^{\bar{\Lambda}^{\mathrm{T}}s}\, \mathrm{d}s \geqslant \int_{t-\delta}^t \mathrm{e}^{\bar{\Lambda}s}\bar{H}.\bar{H}^{\mathrm{T}}\mathrm{e}^{\bar{\Lambda}^{\mathrm{T}}s}\, \mathrm{d}s$$

$$= \mathrm{e}^{\bar{\Lambda}(t-\delta)}\left(\int_0^\delta \mathrm{e}^{\bar{\Lambda}s}\bar{H}.\bar{H}^{\mathrm{T}}\mathrm{e}^{\bar{\Lambda}^{\mathrm{T}}s}\, \mathrm{d}s\right)\mathrm{e}^{\bar{\Lambda}^{\mathrm{T}}(t-\delta)}$$

$$= \lambda_{\min}\left(\int_0^\delta \mathrm{e}^{\bar{\Lambda}s}\bar{H}.\bar{H}^{\mathrm{T}}\mathrm{e}^{\bar{\Lambda}^{\mathrm{T}}s}\, \mathrm{d}s\right)\mathrm{e}^{\bar{\Lambda}(t-\delta)}\mathrm{e}^{\bar{\Lambda}^{\mathrm{T}}(t-\delta)} \tag{56}$$

Notice that (Λ, H) is controllable and hence $(\bar{\Lambda}, \bar{H})$ is controllable. Therefore

$$\lambda_{\min}\left(\int_0^\delta \mathrm{e}^{\bar{\Lambda}s}\bar{H}.\bar{H}^{\mathrm{T}}\mathrm{e}^{\bar{\Lambda}^{\mathrm{T}}s}\, \mathrm{d}s\right) > 0 \tag{57}$$

Set

$$\Sigma_{t-\delta} = \begin{bmatrix} 1 & & & \\ t-\delta & & \ddots & \\ \vdots & & \ddots & \ddots \\ (t-\delta)^{p+q-1}/(p+q-1)! & \dots & t-\delta & 1 \end{bmatrix}.$$

Then from (53) we have

$$\mathrm{e}^{\bar{\Lambda}(t-\delta)} = \mathrm{e}^{\alpha(t-\delta)}\Sigma_{t-\delta}. \tag{58}$$

It is easy to see that

$$\det\left(\sum\nolimits_{t-\delta} \sum\nolimits_{t-\delta}^{\mathrm{T}}\right) = 1 \quad \text{and} \quad \lambda_{\max}\left(\sum\nolimits_{t-\delta} \sum\nolimits_{t-\delta}^{\mathrm{T}}\right) \leqslant (p+q) \sum_{i=0}^{p+q-1} (t-\delta)^{2i}$$

where $\lambda_{\max}(X)$ denotes the maximum eigenvalue of X.

Thus from the fact that

$$\det(X) = \prod_{i=1}^{p+q} \lambda_i(X)$$

for any $(p+q)\times(p+q)$ matrix with eigenvalues $\lambda_i(X)$ $(i = 1, \dots, p+q)$ it follows that

$$\lambda_{\min}\left(\sum\nolimits_{t-\delta} \sum\nolimits_{t-\delta}^{\mathrm{T}}\right) \geqslant \left[\lambda_{\max}\left(\sum\nolimits_{t-\delta} \sum\nolimits_{t-\delta}^{\mathrm{T}}\right)\right]^{-(p+q-1)} \geqslant \left((p-q)\sum_{i=0}^{p+q-1}(t-\delta)^{2i}\right)^{-(p+q-1)}$$

$$\geqslant (p+q)^{-2(p+q-1)}(t-\delta)^{-2(p+q-1)^2} \quad \forall t \geqslant 1+\delta$$

From this and (58) we obtain

$$\lambda_{\min}(\mathrm{e}^{\bar{\Lambda}(t-\delta)}\mathrm{e}^{\bar{\Lambda}^{\mathrm{T}}(t-\delta)}) \geqslant \mathrm{e}^{2\alpha(t-\delta)}(p+q)^{-2(p+q-1)}(t-\delta)^{-2(p+q-1)^2} \quad \forall t \geqslant 1+\delta$$

which together with $\alpha > 0$, (57) and (56) implies the desired result (55). Therefore (52) is true.

We are now in a position to prove (19).

Let

$$U_t = [u_t, Su_t, \dots, S^{p+q-1}u_t]^{\mathrm{T}} \quad \forall t \geqslant 0$$

$$W_t = y_0 + C(S)w_t + S\eta_t, \qquad M = [M_1, M_2]^{\mathrm{T}}$$

with

$$M_1^{\mathrm{T}} \triangleq \left.\overbrace{\begin{pmatrix} 0 & b_1 & \dots & \dots & \dots & \dots & b_q & 0 & \dots & 0 \\ 0 & 0 & \ddots & & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \ddots & 0 \\ 0 & \dots & 0 & 0 & b_1 & \dots & \dots & \dots & \dots & b_q \end{pmatrix}}^{p+q}\right\} p$$

$$M_2^{\mathrm{T}} \triangleq \left.\overbrace{\begin{pmatrix} 1 & a_1 & \dots & \dots & \dots & \dots & a_p & 0 & \dots & 0 \\ 0 & 1 & \ddots & & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \ddots & 0 \\ 0 & \dots & 0 & 1 & a_1 & \dots & \dots & \dots & \dots & a_p \end{pmatrix}}^{p+q}\right\} q$$

Then from (1) it follows that

$$A(S)\varphi_t = MU_t + [W_t, SW_t, \ldots, S^{p-1}W_t, \underbrace{0, \ldots, 0}_{q}]^{\mathrm{T}}$$

which implies that

$$\lambda_{\min}\left(\int_0^t (A(S)\varphi_s)(A(S)\varphi_s)^{\mathrm{T}}\,\mathrm{d}s\right) \geqslant \frac{1}{2}\lambda_{\min}\left(M\int_0^t U_sU_s^{\mathrm{T}}\,\mathrm{d}sM^{\mathrm{T}}\right) - \rho_2\sum_{i=0}^{p+l-1}\int_0^t (S^iw_s)^2\,\mathrm{d}s - \rho_2\sum_{i=0}^{p}\int_0^t (S^i\eta_s)^2\,\mathrm{d}s - \rho_2\sum_{i=0}^{p-1} t^{2i+1} \tag{59}$$

where ρ_2 is a positive constant.

By an argument similar to (48) we have

$$\int_0^t (S^i\eta_s)^2\,\mathrm{d}s \leqslant \int_0^t \frac{t^{2i}-s^{2i}}{2i(2i-1)[(i-1)!]^2}\,\eta_s^2\,\mathrm{d}s \leqslant \rho_2'(t+1)^{2i+1} \quad \text{a.s.} \tag{60}$$

where $i = 0, 1, 2, \ldots$ and ρ_2' is a positive constant.

Similarly

$$\lambda_{\min}\left(\int_0^t (A(S)\varphi_s)(A(S)\varphi_s)^{\mathrm{T}}\,\mathrm{d}s\right) = \min_{\|x\|=1}\int_0^t \left|\sum_{i=0}^{p} a_iS^ix^{\mathrm{T}}\varphi_s\right|^2\mathrm{d}s$$

$$\leqslant \min_{\|x\|=1} \rho_3\sum_{i=0}^{p} t^{2i+1}\int_0^t (x^{\mathrm{T}}\varphi_s)^2\,\mathrm{d}s = \rho_3\sum_{i=0}^{p} t^{2i+1}\lambda_{\min}\left(\int_0^t \varphi_s\varphi_s^{\mathrm{T}}\,\mathrm{d}s\right) \tag{61}$$

where ρ_3 is a positive constant.

From (48) and (59)–(61) we have

$$\lambda_{\min}\left(\int_0^t \varphi_s\varphi_s^{\mathrm{T}}\,\mathrm{d}s\right) \geqslant \frac{1}{2\rho_3}\,\lambda_{\min}(MM^{\mathrm{T}})\left(\sum_{i=0}^{p} t^{2i+1}\right)^{-1}\lambda_{\min}\left(\int_0^t U_sU_s^{\mathrm{T}}\,\mathrm{d}s\right) - \frac{\rho_2(\varkappa_1+\rho_2'+1)}{\rho_3}\left(\sum_{i=0}^{p} t^{2i+1}\right)^{-1}\sum_{i=0}^{p+l}(t+1)^{2i+1} \quad \text{a.s.} \tag{62}$$

Noticing that for any $t \geqslant \tau$, $u_t = u_t'$, by induction we derive

$$\int_0^t (S^iu_s - S^iu_s')^2\,\mathrm{d}s \leqslant \left(\frac{1}{2}t^2\right)^i \int_0^\tau (u_s-u_s')^2\,\mathrm{d}s \quad \forall t \geqslant \tau, \quad i = 0, 1, \ldots \tag{63}$$

Thus for any $x \in \mathbb{R}^{p+q}$ with $\|x\| = 1$ we have

$$\int_0^t (x^{\mathrm{T}}U_s)^2\,\mathrm{d}s \geqslant \frac{1}{2}\int_0^t (x^{\mathrm{T}}U_s')^2\,\mathrm{d}s - \int_0^t \|U_s - U_s'\|^2\,\mathrm{d}s$$

$$\geqslant \frac{1}{2}\int_0^t (x^{\mathrm{T}}U_s')^2\,\mathrm{d}s - (p+q)\sum_{i=0}^{p+q-1}\int_0^t (S^iu_s - S^iu_s')^2\,\mathrm{d}s$$

$$\geqslant \frac{1}{2}\int_0^t (x^{\mathrm{T}}U_s')^2\,\mathrm{d}s - (p+q)\sum_{i=0}^{p+q-1}\left(\frac{1}{2}t^2\right)^i\int_0^\tau (u_s-u_s')^2\,\mathrm{d}s$$

which implies that

$$\lambda_{\min}\left(\int_0^t U_sU_s^{\mathrm{T}}\,\mathrm{d}s\right) \geqslant \frac{1}{2}\,\lambda_{\min}\left(\int_0^t U_s'U_s'^{\mathrm{T}}\,\mathrm{d}s\right) - (p+q)\sum_{i=0}^{p+q-1}\left(\frac{1}{2}t^2\right)^i\int_0^\tau (u_s-u_s')^2\,\mathrm{d}s$$

Substituting this into (62) and recalling (52), we obtain the first assertion of Lemma 3.

We now prove the second assertion of the lemma.

From $U_t = P\bar{U}_t$, (54) and (63) it is easy to see that there exist $\rho_4 \geqslant 0$ and $\gamma_0 > 1$ such that

$$\|U_t\|^2 \leqslant \rho_4\gamma_0^t \quad \forall t \geqslant 0 \tag{64}$$

From (1) it follows that

$$Y_t=\begin{bmatrix} -a_1 & \dots & -a_{p-1} & -a_p \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix} SY_t+[SB(S)u_t+W_t,\underbrace{0,\dots,0}_{p-1}]^{\mathrm{T}}$$

This together with (64) and Assumption A2 implies the second assertion of Lemma 3. Q.E.D.

APPENDIX IV

Proof of Lemma 5

We first show that it is impossible that $\tau_i<\infty$ and $\sigma_{i+1}=\infty$ on a set $\mathscr{D}$ of positive probability for an integer-valued random variable $i\geqslant 0$. In fact, if there were a set $\mathscr{D}$ with positive probability, i.e. $P(\mathscr{D})>0$, and for every sample $\omega\in\mathscr{D}$ there were an $i(\omega)\geqslant 0$ (for simplicity we drop ω below) such that $\tau_i<\infty$ and $\sigma_{i+1}=\infty$, then $u_t=u_t'$ for all $t\geqslant\tau_i$. Thus by Lemmas 2 and 3 we would have

$$\|\theta_t-\theta\|^2=O\left(\frac{(t+1)^{2l+1}}{b^t}\right)\quad\text{a.s. on }\mathscr{D}\tag{65}$$

From Lemma 3 of Reference 17 and the fact that $\partial(R(S))\geqslant\partial(C(S))+1$ it follows that

$$\int_0^t\left(\frac{C(S)}{R(S)}w_s\right)^2\mathrm{d}s=O(t)\quad\text{a.s.}$$

while from (1), (6), (8), (20) and (21) it follows that

$$\xi_s^y-\theta_t^{\mathrm{T}}\zeta_s=(\theta-\theta_t)^{\mathrm{T}}\zeta_s+\frac{C(S)}{R(S)}w_s+\frac{S}{R(S)}\eta_s+\frac{1}{R(S)}y_0\quad\forall t,\quad s\geqslant 0$$

Therefore by Assumption A2, Lemma 1 and (25) we find that

$$\begin{aligned}&\frac{1}{\Gamma_t((t+1)^2)}\int_0^t(\xi_s^y-\theta_t^{\mathrm{T}}\zeta_s)^2\,\mathrm{d}s\\ &\leqslant\frac{4}{\Gamma_t((t+1)^2)}\left[\int_0^t[(\theta-\theta_t)^{\mathrm{T}}\zeta_s]^2\,\mathrm{d}s+\int_0^t\left(\frac{C(S)}{R(S)}w_s\right)^2\mathrm{d}s+\int_0^t\left(\frac{S}{R(S)}\eta_s\right)^2\mathrm{d}s+\int_0^t\left(\frac{1}{R(S)}y_0\right)^2\mathrm{d}s\right]\\ &=O\left(\|\theta_t-\theta\|^2+\frac{1}{(t+1)^2}\right)=O\left(\frac{1}{(t+1)^2}\right)\quad\text{a.s. on }\mathscr{D}\end{aligned}\tag{66}$$

where (65) is invoked for the last inequality.

From (66), by (22) we conclude that there exists a random integer $t_1\geqslant 0$ such that for any $t\geqslant t_1$

$$\frac{1}{\Gamma_t((t+1)^2)}\int_0^t(\xi_s^y-\theta_t^{\mathrm{T}}\zeta_s)^2\,\mathrm{d}s\leqslant\varepsilon_t^2\quad\text{a.s. on }\mathscr{D}\tag{67}$$

From (67), (65) and Lemma 3 we conclude that $\sigma_{i+1}<\infty$ a.s. on $\mathscr{D}$. This contradicts that $\sigma_{i+1}=\infty$ on $\mathscr{D}$ and $P(\mathscr{D})>0$.

We now prove that $\tau_i=\infty$ a.s. for some integer-valued random variable $i\geqslant 1$.

From Lemma 2 it follows that

$$\|\theta_{\sigma_i}-\theta\|^2=O\left(\frac{(\sigma_i+1)^{2l+1}+\log r_{\sigma_i}}{\lambda_{\min}^{(\sigma_i)}}\right)\quad\text{a.s.}$$

which, incorporating $\varrho=\partial(R(S))\geqslant\partial(C(S))+1=l+1$ and the definition of σ_i, implies that

$$\|\theta_{\sigma_i}-\theta\|^2=O\left(\frac{\varepsilon_{\sigma_i}^2}{\sigma_i}\right)\quad\text{a.s.}\tag{68}$$

Similarly to (66) we have

$$\frac{1}{\Gamma_t((\sigma_i+1)^2)}\int_0^t(\xi_s^y-\theta_{\sigma_i}^{\mathrm{T}}\zeta_s)^2\,\mathrm{d}s=O\left(\|\theta_{\sigma_i}-\theta\|^2+\frac{1}{(\sigma_i+1)^2}\right)\leqslant\varepsilon_{\sigma_i}^2\quad\text{a.s.}\tag{69}$$

where the last inequality is valid for some large enough i and any $t\geqslant\sigma_i$ because of (68) and (22).

Hence there must be a $\tau_i=\infty$ a.s. for some $i\geqslant 1$ possibly depending upon ω. Q.E.D.

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